# Digital Filtering Technique for the FDTD Implementation of the Anisotropic Perfectly Matched Layer

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***Abstract*—A new algorithm is presented for the finite difference time domain (FDTD) implementation of the anisotropic perfectly matched layer (PML) using the digital signal processing. The algorithm is based on modeling the anisotropic PML region as a set of infinite-impulse response (IIR) digital filters. The advantage of the proposed method is that it allows direct FDTD implementation of Maxwell's equations in the PML region. In addition, the formulations are implemented using $\mathrm{D}$ and $\mathrm{B}$ fields rather than $\mathrm{E}$ and $\mathrm{H}$, and this allows the PML to be independent from the material properties of the FDTD computational domains. Numerical tests have been carried out in two dimensions to validate the formulations.**

***Index Terms*—Anisotropic perfectly matched layer, digital filters, digital signal processing, finite difference time domain.**

## I. INTRODUCTION

THE perfectly matched layer (PML), introduced by Berenger [1], has been shown to be the most popular finite-difference time domain (FDTD ) [2] absorbing boundary condition. As originally proposed, Berenger's PML is based on splitting the field components and can only be used for truncating lossless media. For lossy media, alternative PML formulations have been introduced [3]–[5]. Among these formulations, the anisotropic PML [3] has the advantage of maintaining Maxwell's equations in their familiar form. Different techniques have been developed for implementing the anisotropic PML in the FDTD method without the need for Berenger's field splitting [6], [7].

In this letter, a new and simple method is presented for the FDTD implementation of the anisotropic PML using the digital signal processing (DSP). The method, named as digital filter PML (DF-PML), is based on modeling the anisotropic PML region as a set of infinite impulse response (IIR) digital filters. The advantage of the proposed method is that it allows direct FDTD implementation of Maxwell's equations in the PML region. In addition, the proposed formulations are implemented using $\mathbf{D}$ and $\mathbf{B}$ fields rather than the conventional $\mathbf{E}$ and $\mathbf{H}$. This makes the formulations to be independent from the material properties of the FDTD computational domains [5]. It should be mentioned that the proposed DF-PML formulations differ from those in [8], which is based on incorporating the DSP into the stretched coordinate PML [4], in the fact that the DF-PML applies the digital filtering technique between $\mathbf{E}$ and $\mathbf{D}$ (or $\mathbf{H}$ and $\mathbf{B}$) rather than between the stretched coordinate variables [4] and the spatial derivatives of $\mathbf{E}$ (or $\mathbf{H}$) as mentioned in [8]. Two-dimensional numerical tests have been carried out to validate the proposed formulations.

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## II. FORMULATION

In the anisotropic PML region [6], the frequency domain Maxwell's equations can be written as

$$\nabla \times \mathbf{H} = j\omega\varepsilon_o \hat{\varepsilon}_r(\omega)\bar{\varepsilon}(\omega)\mathbf{E} \tag{1}$$
$$\nabla \times \mathbf{E} = -j\omega\mu_0 \hat{\mu}_r(\omega)\bar{\varepsilon}(\omega)\mathbf{H} \tag{2}$$

where $\hat{\varepsilon}_r(\omega)$ and $\hat{\mu}_r(\omega)$ are, respectively, the relative permittivity and permeability of the FDTD computational domain and $\bar{\varepsilon}(\omega)$ is defined [6] as

$$\bar{\varepsilon}(\omega) = \begin{bmatrix} \frac{S_y S_z}{S_x} & & \\ & \frac{S_x S_z}{S_y} & \\ & & \frac{S_x S_y}{S_z} \end{bmatrix}, S_\eta = 1 + \frac{\sigma_\eta}{j\omega\varepsilon_o}, \ (\eta = x, y, z) \tag{3}$$

where $\sigma_\eta, (\eta = x, y, z)$ are the conductivity profiles of the PML region in the $\eta$–coordinates. Equations (1) and (2) can be written in terms of $\mathbf{D}$ and $\mathbf{B}$ fields as

$$\nabla \times \mathbf{H} = j\omega \hat{\varepsilon}_r(\omega)\mathbf{D} \tag{4}$$
$$\nabla \times \mathbf{E} = -j\omega \hat{\mu}_r(\omega)\mathbf{B} \tag{5}$$

where $\mathbf{D}$ and $\mathbf{B}$ are given by

$$\mathbf{D} = \varepsilon_o \bar{\varepsilon}(\omega)\mathbf{E} \tag{6}$$
$$\mathbf{B} = \mu_0 \bar{\varepsilon}(\omega)\mathbf{H}. \tag{7}$$

In these formulations, $\mathbf{D}$ and $\mathbf{B}$ are obtained easily through discretizing (4) and (5) by following Yee's algorithm [2]. To obtain $\mathbf{E}$ from $\mathbf{D}$ using (6) (or $\mathbf{H}$ from $\mathbf{B}$ using (7)), the following digital filtering technique is proposed. As an example, consider the $E_z$ field component of (6)

$$D_z = \varepsilon_o \varepsilon_{zz}(\omega) E_z \tag{8}$$

where

$$\varepsilon_{zz}(\omega) = \frac{S_x S_y}{S_z} = \frac{\left(1 + \frac{\sigma_x}{j\omega\varepsilon_o}\right)\left(1 + \frac{\sigma_y}{j\omega\varepsilon_o}\right)}{\left(1 + \frac{\sigma_z}{j\omega\varepsilon_o}\right)}. \tag{9}$$

Transforming (8) into the $Z$–domain, we obtain

$$D_z(z) = \varepsilon_o \varepsilon_{zz}(z) E_z(z) \tag{10}$$

where $\varepsilon_{zz}(z)$ is the $Z$–transform of $\varepsilon_{zz}(\omega)$ which can be modeled as a second order IIR digital filter by transforming (9) into the $s$–domain, using the relation $j\omega \longrightarrow s$, as

$$\varepsilon_{zz}(s) = \frac{\left(s + \frac{\sigma_x}{\varepsilon_o}\right)\left(s + \frac{\sigma_y}{\varepsilon_o}\right)}{s\left(s + \frac{\sigma_z}{\varepsilon_o}\right)} \tag{11}$$

and applying the bilinear transformation method [9] to (11) using the relation $s = 2(1 - z^{-1})/\Delta t(1 + z^{-1})$, where $\Delta t$ is the sampling time step, and after some manipulations $\varepsilon_{zz}(z)$ can be written as

$$\varepsilon_{zz}(z) = \beta_x \frac{\left(1 - \alpha_x z^{-1}\right)}{\left(1 - z^{-1}\right)} \frac{\beta_y}{\beta_z} \frac{\left(1 - \alpha_y z^{-1}\right)}{\left(1 - \alpha_z z^{-1}\right)} \tag{12}$$

where $\beta_\eta = 1 + \Delta t \sigma_\eta / 2\varepsilon_o$ ($\eta = x, y, z$) and $\alpha_\eta = (1 - \Delta t \sigma_\eta / 2\varepsilon_o) / (1 + \Delta t \sigma_\eta / 2\varepsilon_o)$. Clearly, (12) represents first order cascade realization [9] of a second order IIR digital filter of the form

$$\varepsilon_{zz}(z) = C \frac{b_{0z} + b_{1z} z^{-1} + b_{2z} z^{-2}}{1 + a_{1z} z^{-1} + a_{2z} z^{-2}} \tag{13}$$

where $C = \beta_x \beta_y / \beta_z$ and the filter coefficients are given as: $a_{1z} = -(1 + \alpha_z)$, $a_{2z} = \alpha_z$, $b_{0z} = 1$, $b_{1z} = -(\alpha_x + \alpha_y)$ and $b_{2z} = \alpha_x \alpha_y$. Substituting (12) into (10), we obtain

$$D_z(z) = \varepsilon_o \beta_x \frac{\left(1 - \alpha_x z^{-1}\right)}{\left(1 - z^{-1}\right)} \frac{\beta_y}{\beta_z} \frac{\left(1 - \alpha_y z^{-1}\right)}{\left(1 - \alpha_z z^{-1}\right)} E_z(z). \tag{14}$$

To write (14) in FDTD form, we introduce the variable

$$f_z(z) = \frac{\beta_y}{\beta_z} \frac{\left(1 - \alpha_y z^{-1}\right)}{\left(1 - \alpha_z z^{-1}\right)} E_z(z) \tag{15}$$

then, (14) can be written as

$$D_z(z) = \varepsilon_o \beta_x \frac{\left(1 - \alpha_x z^{-1}\right)}{\left(1 - z^{-1}\right)} f_z(z). \tag{16}$$

Therefore, $E_z$ can be computed from (15) as

$$E_z(z) = \alpha_y z^{-1} E_z(z) + \frac{\beta_z}{\beta_y}\left(1 - \alpha_z z^{-1}\right) f_z(z) \tag{17}$$

where $f_z(z)$ can be obtained from (16) as

$$f_z(z) = \alpha_x z^{-1} f_z(z) + \frac{1}{\varepsilon_o \beta_x}\left(1 - z^{-1}\right) D_z(z). \tag{18}$$

As the $z^{-1}$ operator in the $Z$–domain corresponds to a delay of one time step in the sampled time domain [9], (17) and (18) can be written in FDTD form, respectively, as

$$E_z^{n+1} = \alpha_y E_z^n + \frac{\beta_z}{\beta_y}\left(f_z^{n+1} - \alpha_z f_z^n\right) \tag{19}$$

$$f_z^{n+1} = \alpha_x f_z^n + \frac{1}{\varepsilon_o \beta_x}\left(D_z^{n+1} - D_z^n\right) \tag{20}$$

where $E_z^{n+1}$, $D_z^{n+1}$, $f_z^{n+1}$ and the filter coefficients are evaluated at the corresponding Yee's grid position [2]. It should be mentioned that $D_z^{n+1}$ is computed through discretizing (4) following Yee's algorithm [2]. In addition, $E_z^{n+1}$ and $f_z^{n+1}$ can be updated efficiently without storing, respectively, $f_z^n$ and $D_z^n$ in separate arrays. This can be done either by using the two step technique mentioned in [10] or by using two simple temporary variables to store the value of $D_z^n$ and $f_z^n$ [11]. Therefore, the FDTD implementation of (14) requires only one additional auxiliary variable ($f_z$) per FDTD cell. Similar equations can be obtained for the other $E_x$ and $E_y$ field components.

The above formulations are applied in the PML regions where all $S_\eta$, ($\eta = x, y, z$), in (3) overlap, such as the corner PML regions [6]. In the face and edge PML regions [6], simpler formulations can be obtained. As an example, to achieve perfect absorption for waves propagating in the $z$–direction, the elements of (3) should be chosen in the $z$–face PML region as [6]

$$\varepsilon_{\eta\eta}(\omega) = \begin{cases} S_z, & \eta = x, y \\ \frac{1}{S_z}, & \eta = z \end{cases} \tag{21}$$

which can be modeled as a set of first order IIR digital filters as

$$\varepsilon_{\eta\eta}(z) = \begin{cases} \beta_z \frac{\left(1 - \alpha_z z^{-1}\right)}{\left(1 - z^{-1}\right)}, & \eta = x, y \\ \frac{1}{\beta_z} \frac{\left(1 - z^{-1}\right)}{\left(1 - \alpha_z z^{-1}\right)}, & \eta = z \end{cases}. \tag{22}$$

Using (22) and the $Z$–transform of (6), $\mathbf{E}$ can be obtained from $\mathbf{D}$ as

$$E_\eta(z) = \begin{cases} \alpha_z z^{-1} E_\eta(z) + \frac{1}{\varepsilon_o \beta_z}\left(1 - z^{-1}\right) D_\eta(z), & \eta = x, y \\ z^{-1} E_\eta(z) + \frac{\beta_z}{\varepsilon_o}\left(1 - \alpha_z z^{-1}\right) D_\eta(z), & \eta = z \end{cases}. \tag{23}$$

Equation (23) can be written in the FDTD form as

$$E_\eta^{n+1} = \begin{cases} \alpha_z E_\eta^n + \frac{1}{\varepsilon_o \beta_z}\left(D_\eta^{n+1} - D_\eta^n\right), & \eta = x, y \\ E_\eta^n + \frac{\beta_z}{\varepsilon_o}\left(D_\eta^{n+1} - \alpha_z D_\eta^n\right), & \eta = z \end{cases} \tag{24}$$

where $\mathbf{E}$, $\mathbf{D}$ and the filter coefficients are evaluated at the corresponding Yee's grid position [2]. As mentioned, by using [10] or [11], $E_\eta^{n+1}$, ($\eta = x, y, z$), can be updated efficiently without storing $D_\eta^n$, and hence no additional auxiliary variables are needed in the face PML regions. Similar expressions can be obtained in the other face and edge PML regions. To compute $\mathbf{H}$ from $\mathbf{B}$ using (7), it is only required to modify the coefficient of the digital filter described in (12) to include the half space cell offset that exists between $\mathbf{E}$ and $\mathbf{H}$ as described in Yee's algorithm [2].

It is interesting to note that as $\widehat{\varepsilon}_r(\omega)$ and $\widehat{\mu}_r(\omega)$ do not appear in the above formulations, the proposed DF-PML formulations are therefore independent from the material properties of the FDTD computational domain. Hence, for general media, the above formulations can be used without any modification and it is only required to discretize (4) and (5), accordingly.

## III. Numerical Study

In order to validate the proposed formulations, the numerical experiment of [12] was carried out in two dimensions for the TM case. A point source is used to excite $100\Delta x \times 50\Delta y$ isotropic and homogeneous FDTD computational domain at its center. The space cell size in the $x$ and $y$ directions are chosen as $\Delta x = \Delta y = 1.5$ cm and the time step was $\Delta t = 25$ ps. The

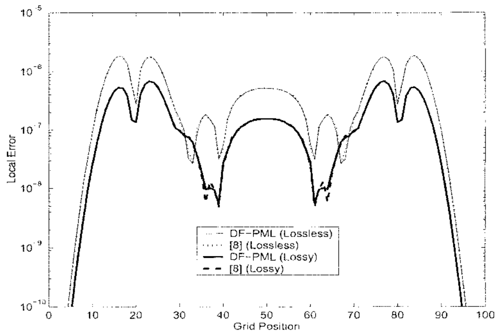


Fig. 1. Local error for the PML/ FDTD computational domain interface along the line $(x, -25\Delta y)$ as observed at time $100\Delta t$ for PML[8, 3, 0.0001%] and for lossless ($\varepsilon_r = \mu_r = 1,\ \sigma = 0.0$) and lossy ($\varepsilon_r = \mu_r = 1,\ \sigma = 0.01$) FDTD domains.

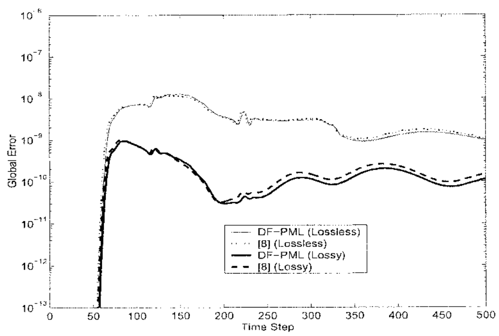


Fig. 2. Global error in the computational domain for PML[8, 3, 0.0001%] and for lossless ($\varepsilon_r = \mu_r = 1,\ \sigma = 0.0$) and lossy ($\varepsilon_r = \mu_r = 1,\ \sigma = 0.01$) FDTD domains.

excitation used is similar to the derivative of the pulse used in [12]. This pulse was preferred as it has low numerical grid dispersion [13]. The reference FDTD solution, having no reflection errors form the domain boundaries, is calculated using a much larger computational domain ($400\Delta x \times 400\Delta y$).

The performance of the proposed DF-PML formulations was investigated for lossless and lossy FDTD computational domains. For both cases, the computational domain was terminated by 8 PML layers backed by a perfect electric conductor (PEC). The PML parameters were chosen to give good absorbing performance and taken as PML [8, 3, 0.0001%], as defined in Berenger's notation [1]. Figs. 1 and 2 show the local and the global errors, as defined in [12], for lossless ($\varepsilon_r = \mu_r = 1,\ \sigma = 0.0$) and lossy ($\varepsilon_r = \mu_r = 1,\ \sigma = 0.01$) FDTD computational domains. The local error was calculated for the PML/FDTD computational domain interface along the line $(x, -25\Delta y)$ as observed at time $100\Delta t$. The results are obtained using the proposed DF-PML formulations and the formulations presented in [8]. As it can be observed from these results, both methods give almost the same error performance. It should be mentioned, however, that the proposed DF-PML formulations require less computational resources than the formulations in [8]. For the test cases, the DF-PML formulations require only one additional auxiliary variable per field component per cell in the corner PML regions and no additional auxiliary variables in the edge PML regions, while the formulations in [8] require two additional auxiliary variables per field component per cell in the corner PML regions and one additional auxiliary variable per field component per cell in the face PML regions. Therefore, significant savings in the memory storage and the computational time requirements can be realized using the DF-PML formulations.

It should be noted that the DF-PML formulations can also be extended for truncating more generalized media such dispersive or anisotropic media without any special treatments. In these cases, all that is needed is to discretize (4) and (5), accordingly.

## IV. Conclusion

In this letter, a new method, DF-PML, which incorporates the DSP into the FDTD implementation of the anisotropic PML is presented for truncating FDTD domains. The method is based on modeling the anisotropic PML region as a set of IIR digital filters. The advantage of the method is that it allows direct FDTD implementation of Maxwell's equations in the PML region. In addition, the formulations are implemented using $\mathbf{D}$ and $\mathbf{B}$ fields rather than $\mathbf{E}$ and $\mathbf{H}$, which allows the formulations to be independent from the material properties of the FDTD computational domain. Numerical tests show that the error performance of the DF-PML is similar to other PML formulations.

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